Dependence of Structure of Flow Simulation Results on Random Errors

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Abstract

In the present paper, we try to investigate the dependencies of the structure of numerical solutions of the incompressible Navier-Stokes equations on insertion of random errors. The effect of randomness in determining the asymptotic structure of numerical results is studied by using the nonlinear dynamics approach. The incompressible Navier-Stokes equations and the continuity equation are solved numerically by using the MAC(Maker-And-Cell) method and implicit temporal scheme. The model adopted in the present study is a flow around a two-dimensional circular cylinder and the Reynolds number is 1500. Not only effects of explicitly added randomness, but also those of random errors inserted in solving the Poisson equation for the pressure under the rough condition for convergence, are discussed. It is shown that the dependencies of asymptotic numerical solutions on the amplitude of randomness are very sensitive.
Keywords : random errors, incompressible Navier-Stokes equations, Numerical simulation, Asymptotic behavior of numerical solutions

1 Introduction

Recently, analyses of fluid motions by the numerical simulation technique have been one of the standard tools. However, several types of errors often induce the appearance of ghost solutions which do not correspond to the true ones of original differential equations. For example, the truncation error which becomes large under the selection of large time increment value makes the calculation system unstable and we often get very complicated flow structure even in the case of low Reynolds number. As a result, qualitative interpretation of a given solution becomes more and more difficult. These ghost solutions for nonlinear differential equations have been studied in detail analytically and numerically.[1)~8)] We also studied characteristics of asymptotic numerical solutions of simple scalar nonlinear differential equations and practical fluid equations. On the other hand, we must solve the Poisson equations for the pressure if the MAC(Marker And Cell) method[9] are adopted in the direct simulation for incompressible fluid motions. Then it is necessary to solve the simultaneous equations. Though direct method such as the Gaussian elimination method are used for solving moderately sized linear systems, it is a matter of course to use the iterative methods such as Gauss-Seidel method in which the condition for convergence is determined by norm of error(α) for the large sized ones. Needless to say, the structure of numerical solution depends on the condition or criterion for convergence in solving simultaneous equations. Random errors inserted by selecting large α values induce the appearance of stable spurious solutions.[10] In particular, the structure of flow field may be affected by such random errors in the early stage of calculations. Therefore, it is very important to discuss whether the amplitude of such random error becomes a parameter to determine the structural instability or not. In the present paper, we try to discuss the dependence of the structure of numerical solutions of incompressible fluid equations on insertion of random errors in solving simultaneous equations. Furthermore, the analogy and dissimilarity between the dependence of the structure of numerical solutions on the condition for convergence and that on the forcibly added randomness given by using the pseudo-random number row are discussed. Here, we use the notation of “ r ” for the amplitude of the forcibly added randomness. In the previous paper[11], we discussed the averaged structure of the system which includes the random noise and clarified the relation between the size of noise and characteristics of obtained numerical solutions of the logistic equation and the Lorenz equations. Furthermore, we discussed the dependence of the structure of numerical solutions on the randomness in the practical fluid simulations. [12] In that study, it was clarified that weak noises make the system change and the effects of the noises are similar to those of the fourth viscosity terms. In the present paper, we adopted a flow around a circular cylinder as the simple model and compare the unsteady structure influenced by random errors in the cases of selection of large “ α ” value and “ r ” ones by comparing trajectories reconstructed from the long period of time series of C_D(drag coefficient).

The numerical scheme used in the present paper and conditions of computations such as the grid systems, boundary conditions and so on are expressed briefly in Section 2. In Section 3, the dependencies of the numerical results of the practical fluid simulations on the condition for convergence in iteration processes and the forcibly added randomness are discussed.

2 Numerical Algorithm and Numerical Conditions

2.1 Basic equations

The non-dimensional incompressible Navier-Stokes equations and the continuity equation are given as follows:

$$\begin{cases} \mathrm{div}\mathbf{V} = 0 \\ \dfrac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \bullet \mathrm{grad})\mathbf{V} = -\mathrm{grad}p + \dfrac{1}{Re}\Delta\mathbf{V} \quad (+ \;\; rRa), \end{cases} \tag{1}$$

where $\mathbf{V} = (u, v)$, p and Re denote velocity vector, pressure and the Reynolds number, respectively. The Reynolds number in the present study is fixed to 1500. The forcibly added randomness(Ra) is given by using pseudo-random number and we consider the parameter " r " which denotes the strength of randomness.

2.2 Numerical algorithm

The Poisson equation for the pressure can be derived on the basis of MAC method.[9)]

$$\Delta p = -\mathrm{div}(\mathbf{V} \bullet \mathrm{grad})\mathbf{V} + D, \tag{2}$$

where

$$D = -\frac{\partial \mathrm{div}\mathbf{V}}{\partial t} + \frac{1}{Re}\Delta\mathrm{div}\mathbf{V}. \tag{3}$$

In the present study, we employed the generalized transformation of coordinates, $(x, y) \rightarrow (\xi, \eta)$, then we get the transformed Poisson equation as follows:

$$\begin{aligned} \Delta p \;\; = \;\; & -\frac{(y_\eta u_\xi - y_\xi u_\eta)^2 + 2(x_\xi u_\eta - x_\eta u_\xi)(y_\eta v_\xi - y_\xi v_\eta) + (x_\xi v_\eta - x_\eta v_\xi)^2}{J^2} \\ & -\frac{y_\eta u_\xi - y_\xi u_\eta + x_\xi v_\eta - x_\eta v_\xi}{J\Delta t}, \end{aligned} \tag{4}$$

where J is the Jacobian of transformation. The Poisson equation is solved using Gauss-Seidel scheme. All spatial derivatives except for those of the nonlinear convection term are discretized by using the central finite difference. For those of the convection terms, we considered parameter ε in order to discuss the effects of fourth order artificial viscosity term based on the third-order upwind schemes,[13)]

$$
\begin{aligned}
f\frac{\partial u}{\partial \xi} \quad = \quad & \frac{f_i(-u_{i+2}+8u_{i-1}-8u_{i+1}+u_{i-2})}{12\Delta\xi} \\
& +\varepsilon\frac{|\,f_i|\,(u_{i+2}-4u_{i-1}+6u_i-4u_{i+1}+u_{i-2})}{4\Delta\xi^4}. \qquad (5)
\end{aligned}
$$

Viscous terms in eq.(1) are discretized by using the second order central difference scheme. For the time marching of the Navier-Stokes equations, the first order backward Euler implicit scheme is employed. In the present implicit scheme, the nonlinear convection term is linearized for $\mathbf{V}^{n+1}$ as follows:

$$
\left(\mathbf{V}^{n+1}\cdot\nabla\right)\mathbf{V}^{n+1} \approx \left(\mathbf{V}^{n}\cdot\nabla\right)\mathbf{V}^{n+1}. \qquad (6)
$$

Simultaneous equations of the linearlized implicit scheme are also solved using Gauss-Seidel method.

2.3 Grid systems

The O-type grid systems are used in all cases. The body surface corresponds to $K=1$, the circle of which radius is equal to 1. Outer flow region corresponds to $K=KMAX$, the circle of which radius is set to from 60 to 70. The mesh points are strongly concentrated in the boundary layer and the minimum spacing normal to the surface of the body is set to be less than $\frac{0.049}{\sqrt{Re}}$. Therefore this grid system is fine enough to resolve the flow structure in the boundary layer.

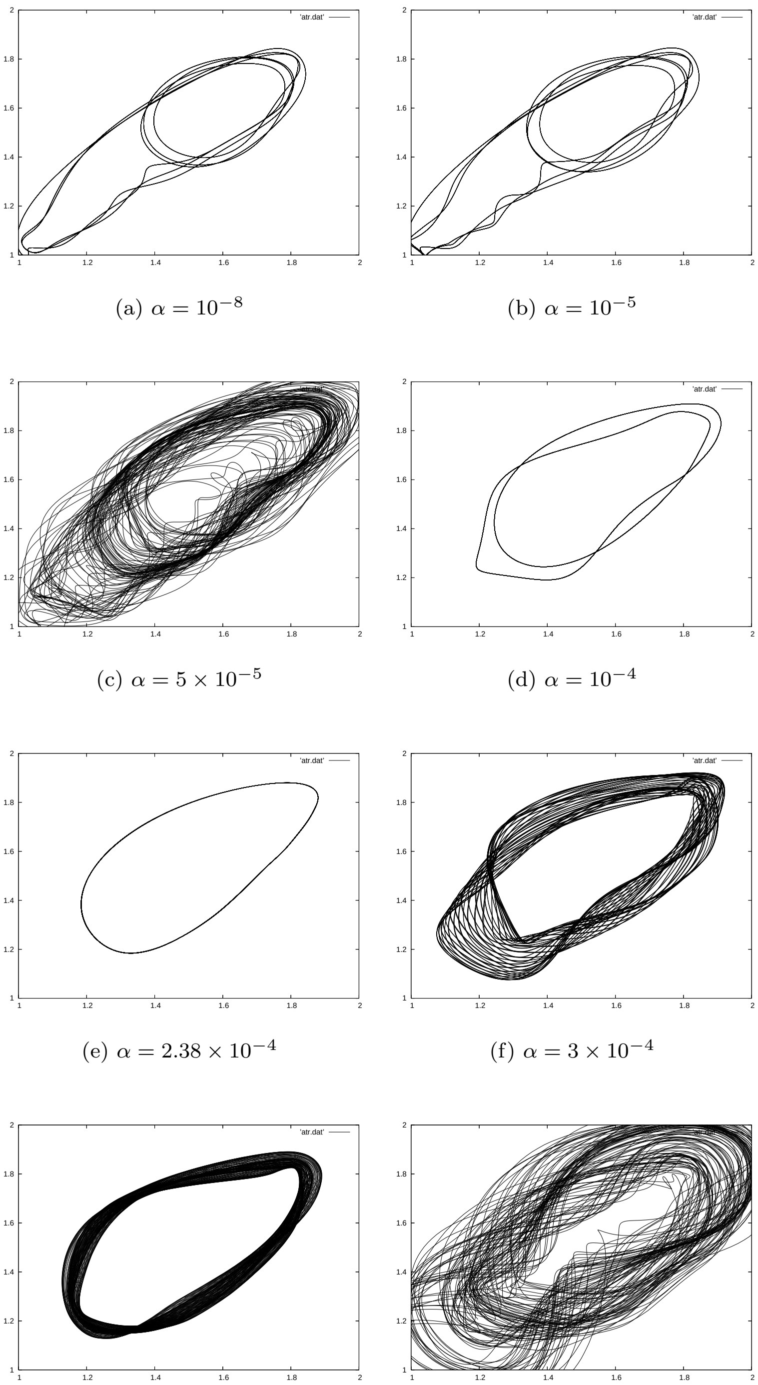
2.4 Boundary conditions

The boundary conditions on the body surface are as follows: The no-slip condition is used for the velocity components. The pressure p along the body surface is obtained by solving a normal momentum equation. At the far boundaries, the free-stream values are specified.

3 Results and Discussion

3.1 Dependence of structure of asymptotic solutions on condition for convergence in solving the Poisson equation for the pressure

In this subsection, we discuss dependence of structure of numerical solutions of fluid simulations on condition for convergence(α) in detail. Figure 1 shows the profiles of the trajectories reconstructed two-dimensionally in the $(C_D(T),\ C_D(T+0.5))$ phase space from time series of C_D in the period



(a) $\alpha = 10^{-8}$

(b) $\alpha = 10^{-5}$

(c) $\alpha = 5 \times 10^{-5}$

(d) $\alpha = 10^{-4}$

(e) $\alpha = 2.38 \times 10^{-4}$

(f) $\alpha = 3 \times 10^{-4}$

(g) $\alpha = 8 \times 10^{-4}$

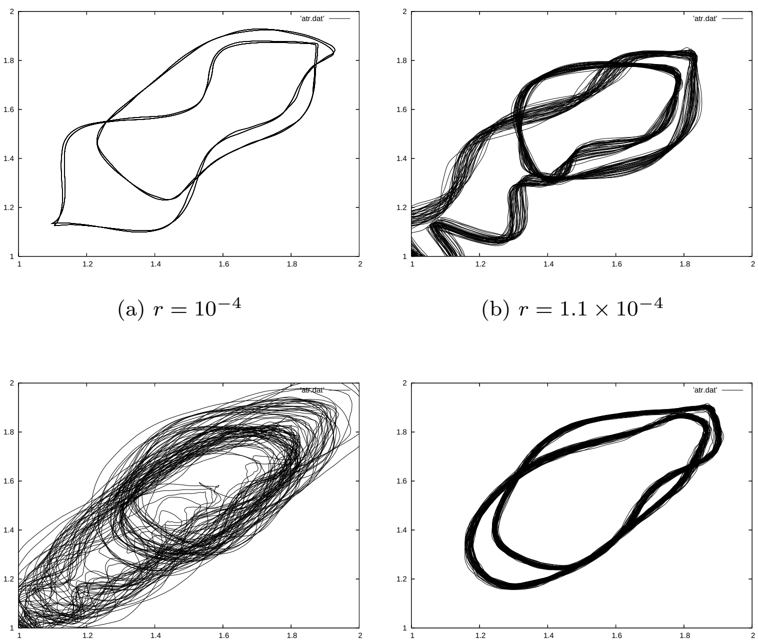
(h) $\alpha = 10^{-3}$

Fig. 1: Comparisons of the trajectories reconstructed on two-dimensional phase-plane from the time series of C_D value.($\varepsilon = 0.17$, $r = 0$)

from T(non-dimensional time)=1500 to 2000. The structure of periodic-6 in the cases of (a) and (b) breaks to the complicated one((c)) and simple periodic structure appears again in the cases of (d) and (e). When α value becomes larger, tori appear((f) and (g)) and very complicated structure can be seen again(h) and calculation finally diverges where α is a little larger than that of (h). This result shows that the insertion of random errors in the large α value cases sometimes makes the structure of asymptotic numerical solution stable. This sequence of change of structure of attractors seems to be similar to the bifurcation process caused by the structural instability in the deterministic dynamical system.

3.2 Effect of the forcibly added randomness and similarity to those of the condition for convergence(α)

In this subsection, we discuss the structure of each sample calculation in which the randomness is forcibly added. Figure 2 shows the sequence of



(a) $r = 10^{-4}$

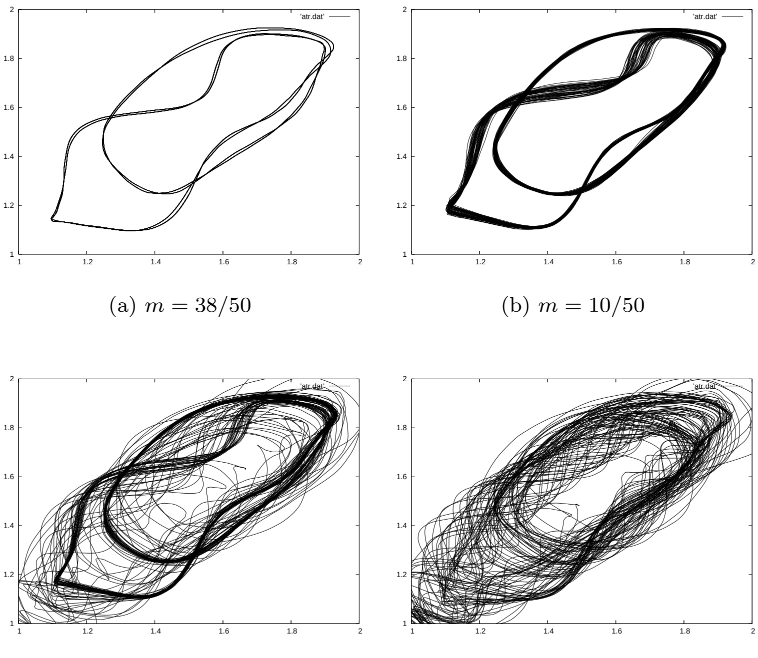
(b) $r = 1.1 \times 10^{-4}$

(c) $r = 1.5 \times 10^{-4}$

(d) $r = 1.8 \times 10^{-4}$

Fig. 2: Comparisons of the trajectories which show the dependence on “r” value.($\varepsilon = 0.17, \alpha = 2.4 \times 10^{-4}$)

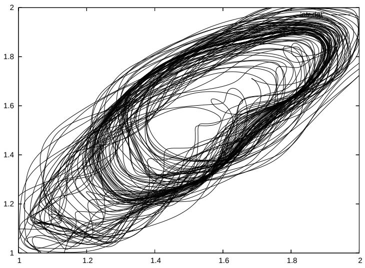
change of structure of attractors by reconstructed two-dimensional trajectories in the case of $\varepsilon = 0.17$ and $\alpha = 2.4 \times 10^{-4}$. The structure of attractor sensitively changes by the amplitude of forcibly added randomness. However, several types of attractors are obtained by using the different randomness. Figure 3 shows calculated attractors in which different seeds of the pseudo-random number row are adopted. Four types attractors are obtained and " m " denotes the ratio of obtained number of samples. We calculated fifty samples for different pseudo-random number row. It is supposed that these differences of types of attractors are determined in the early stage of calculations by the insertion of random errors. Figure 4 shows the typical cases in which the profiles of attractors are similar. It is clear that Fig.4-(a) is similar to Fig.4-(b) and Fig.4-(c) is similar to Fig.4-(d). This result shows that effects of errors inserted in the iteration processes in solving the Poisson equation to the structure of numerical solutions are similar to effects of forcibly added random ones.



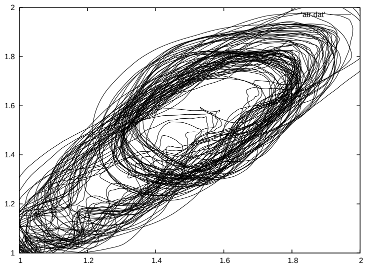
(a) $m = 38/50$ (b) $m = 10/50$

(c) $m = 1/50$ (d) $m = 1/50$

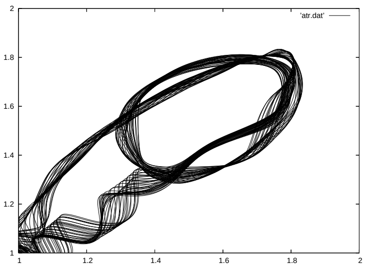
Fig. 3: Types of attractor given by different seeds for random number rows. ($\varepsilon = 0.17, \alpha = 2 \times 10^{-4}, r = 9 \times 10^{-5}$)



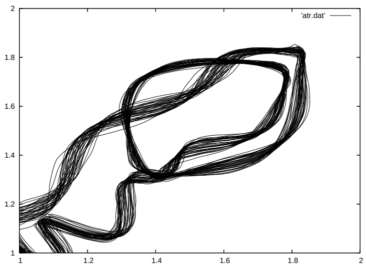
(a) $\alpha = 5 \times 10^{-5}, r = 0$



(b) $\alpha = 2.4 \times 10^{-4}, r = 1.5 \times 10^{-4}$



(c) $\alpha = 5.3 \times 10^{-5}, r = 0$



(d) $\alpha = 2.4 \times 10^{-4}, r = 1.1 \times 10^{-4}$

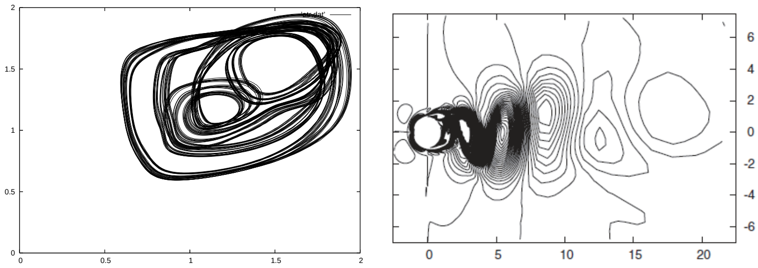
Fig. 4: Profile of attractors showing the similarity of effects.

3.3 Profiles of flow structure in the case of very large α value

In this subsection, we consider the flow structure in the case of very large α value in which the Poisson equation is not adequately solved. Figure 5 is the case of $\alpha = 5 \times 10^{-3}$ and $\varepsilon = 4.3$. Periodical profile can be seen in the trajectory in Fig.5-(a) and vorticity field is not so strange in spite of rough calculation of the Poisson equation for the pressure in Fig.5-(b). This result shows that the global structure of flow field is not so influenced by the condition for convergence of the Poisson equation for the pressure.

4 Summary

In the present paper, we studied dependencies of the numerical fluid dynamics simulation results on the insertion of numerical errors from the viewpoint of randomness. Concretely, we discussed the effect of randomness in



(a) trajectory (b) contour of vorticity

Fig. 5: Profiles of the trajectory and vorticity field in the case of $\alpha = 5\times10^{-3}$ and $\varepsilon = 4.3$.

determining the asymptotic structure of numerical results by using the non-linear dynamics approach. Two types of random errors are considered. One is the randomness given explicitly by adding the pseudo random number row forcibly and another is the random error inserted implicitly in solving the Poisson equation for the pressure under the rough condition for convergence. In both cases, a lot of types of attractors are obtained and sensitive dependencies on the parameter of amplitude of randomness are seen. As a result , insertion of random error has both of stabilizing and destabilizing effects in each case. In any case, it is clear that random errors make the system complicated. In the analysis of non-linear system under the unstable conditions, we cannot completely remove the influence of errors. Therefore, it is necessary to continue the systematic research studied in the present paper.

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